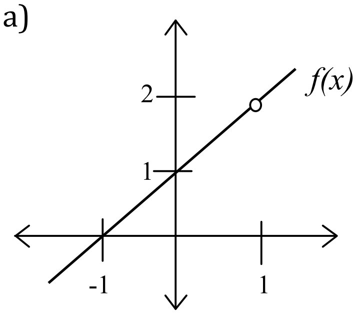
PRECALCULUS: LIMITS AND CONTINUITY

2.1: Limits

Limit notation: $\lim_{x \to a} f(x) = b \leftarrow$ y-value

Example 1: Find the limits graphically.

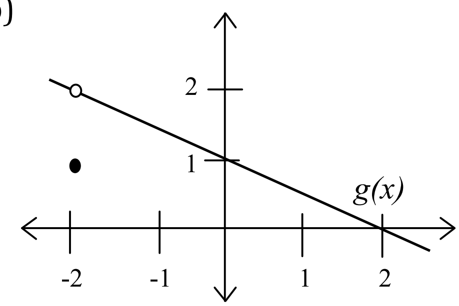
a)



$\lim_{x \to 1} \mathbf{f}(\mathbf{x}) =$ 2

f(1) = DNE

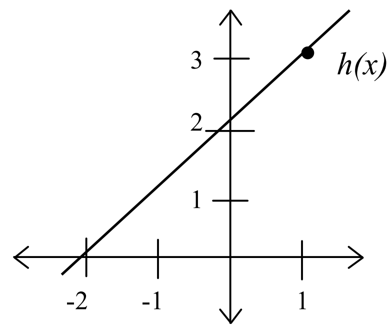
b)



$\lim_{x \to -2} \mathbf{g}(\mathbf{x}) =$ 2

f(-2) = 1

c)



$\lim_{x \to 1} h(x) =$ 3

f(1) = 3

Properties of Limits

If L, M, c, and k are real numbers and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

1. Sum Rule: $\lim_{x \to c} \left(f(x) + g(x) \right) = L + M$
2. Difference Rule: $\lim_{x \to c} \left(f(x) - g(x) \right) = L - M$
3. Product Rule: $\lim_{x \to c} \left(f(x) \cdot g(x) \right) = L \cdot M$
4. Constant Multiple Rule: $\lim_{x \to c} \left(kf(x) \right) = k \cdot L$
5. Quotient Rule: $\lim_{x \to c} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M}, \ M \neq 0$

Example 2: Find the limit algebraically confirm graphically.

a) $\lim_{x \to 2} x^3 + 3x^2 - 6$

$(2)^3 + 3(2)^2 - 6$

$= 8 + 3(4) - 6$

$= 8 + 12 - 6$

$= \boxed{14}$

b) $\lim_{x \to 1} \frac{x^2 + 5x - 4}{x - 3}$

$= \frac{(1)^2 + 5(1) - 4}{1 - 3}$

$= \frac{1 + 5 - 4}{1 - 3} = \frac{2}{-2}$

$= \boxed{-1}$

c) $\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4}$ $\frac{0}{0}$

FACTOR

$\lim_{x \to 2} \frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$

$\lim_{x \to 2} \frac{x+3}{x+2} = \frac{2+3}{2+2} = \boxed{\frac{5}{4}}$

d) $\lim_{x \to -2} \frac{x^3 - 8}{x - 2}$

$= \frac{(-2)^3 - 8}{-2 - 2} = \frac{-8 - 8}{-4}$

$= \frac{-16}{-4} = \boxed{4}$

e) $\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \frac{0}{0}$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 0 & -8 \\ & & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & 0 \end{array}$$

$\lim_{x \to 2} \frac{\cancel{(x-2)}(x^2 + 2x + 4)}{\cancel{x-2}}$

$\lim_{x \to 2} x^2 + 2x + 4$

$= 2^2 + 2(2) + 4$

$= \boxed{12}$

f) $\lim_{x \to -3} \frac{1}{x+3} = \frac{1}{0}$

$\boxed{\text{undefined}}$

D.N.E.

g) $\lim_{x \to 2} \sqrt{x+7} - 5$

$= \sqrt{2+7} - 5$

$= \sqrt{9} - 5$

$= 3 - 5$

$= \boxed{-2}$

Right handed limit: $\lim_{x \to a^+} f(x)$ (from the positive)

Left handed limit: $\lim_{x \to a^-} f(x)$ (from the negative)

Theorem: One-sided and Two-sided Limits

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x)$$

* If the right hand limit equals the left hand limit, the limit exists.

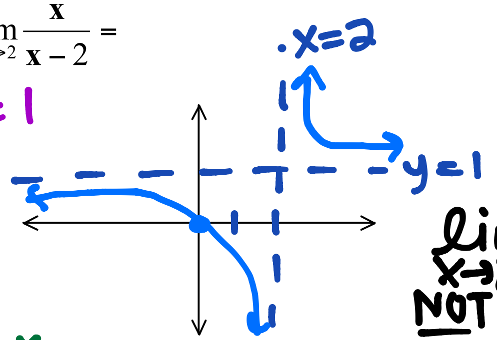
* If the right hand limit does <u>NOT</u> equal the left hand limit, the limit does <u>NOT</u> exist.

Example 3: Use the graph to show that the limit does not exist:

a) $\lim_{x \to 2} \frac{x}{x-2} =$

$\frac{x}{x} = 1$

$\frac{0}{0-2}$



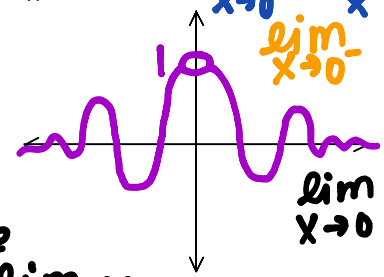
$\lim_{x \to 2^+} \frac{x}{x-2} = \infty$ $\lim_{x \to 2^-} \frac{x}{x-2} = -\infty$

$\lim_{x \to 2} \frac{x}{x-2}$ does NOT exist since $\lim_{x \to 2^+} \frac{x}{x-2} \neq \lim_{x \to 2^-} \frac{x}{x-2}$

b) $\lim_{x \to 0} \frac{\sin x}{x}$

$\lim_{x \to 0^+} \frac{\sin x}{x} = 1$

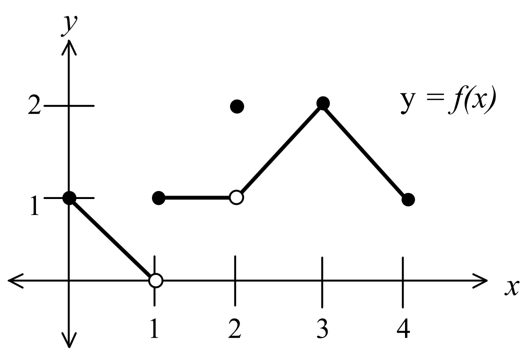
$\lim_{x \to 0^-} \frac{\sin x}{x} = 1$



$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Example 4: Right- and Left-Handed Limits

Use the graph of y = f(x) to find the following limits



At the point	Left-Handed Limit	Right-Handed Limit	Limit
x = 0		$\lim_{x \to 0^+} f(x) = 1$	$\lim_{x \to 0} f(x) = 1$
x = 1	$\lim_{x \to 1^-} f(x) = 0$	$\lim_{x \to 1^+} f(x) = 1$	$\lim_{x \to 1} f(x) =$ DNE
x = 2	$\lim_{x \to 2^-} f(x) = 1$	$\lim_{x \to 2^+} f(x) = 1$	$\lim_{x \to 2} f(x) = 1$
x = 3	$\lim_{x \to 3^-} f(x) = 2$	$\lim_{x \to 3^+} f(x) = 2$	$\lim_{x \to 3} f(x) = 2$
x = 4	$\lim_{x \to 4^-} f(x) = 1$		$\lim_{x \to 4} f(x) = 1$

Example 5: Evaluate the right and left hand limits algebraically. Confirm graphically.

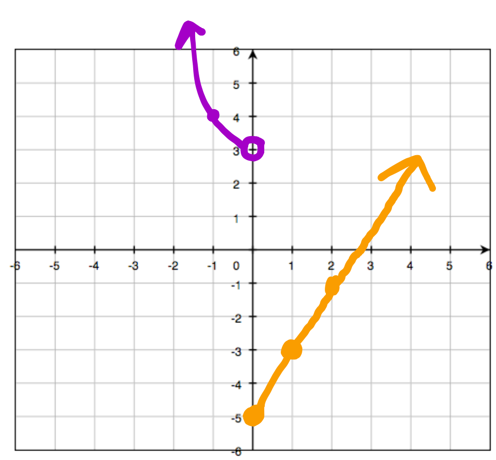
a) $f(x) = \begin{cases} x^2 + 3 & x < 0 \\ 2x - 5 & x \geq 0 \end{cases}$

$\lim_{x \to 0^+} f(x) = 2(0) - 5 = \boxed{-5}$

$\lim_{x \to 0^-} f(x) = 0^2 + 3 = \boxed{3}$

$\lim_{x \to 0} f(x) =$ DNE

$f(0) = 2(0) - 5 = \boxed{-5}$



b) $g(x) = \begin{cases} -2x^2 & x < -2 \\ \underline{3x+1} & \underline{x \geq -2} \end{cases}$

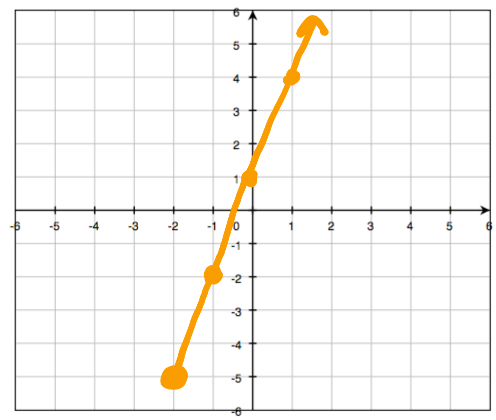
$\lim_{x \to -2^+} g(x) = 3(-2)+1 = \boxed{-5}$

$\lim_{x \to -2^-} g(x) = -2(-2)^2 = -2(4) = \boxed{-8}$

$\lim_{x \to -2} g(x) =$ DNE

$g(-2) = 3(-2)+1 = \boxed{-5}$

$x=2$



c) $h(x) = \begin{cases} x^2 - 2 & x < -3 \\ 2 & -3 \leq x < 1 \\ 2x - 1 & x \geq 1 \end{cases}$

$\lim_{x \to -3^+} h(x) = 2$

$\lim_{x \to -3^-} h(x) = (-3)^2 - 2 = \boxed{7}$

$\lim_{x \to -3} h(x) =$ DNE

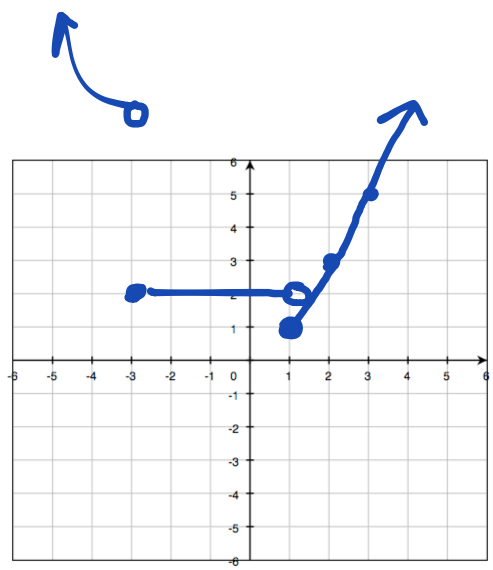
$h(-3) = 2$

$\lim_{x \to 1^+} h(x) = 2(1) - 1 = \boxed{1}$

$\lim_{x \to 1^-} h(x) = 2$

$\lim_{x \to 1} h(x) =$ DNE

$h(1) = 2(1) - 1 = \boxed{1}$



2.2: Limits Involving Infinity

Horizontal Asymptote:	Vertical Asymptote:

Example 1: Finding a Limit as x Approaches Infinity

a) $\lim_{x\to\infty} \frac{\sin x}{x} =$

b) $\lim_{x\to\infty} \frac{3x + \sin x}{x} =$

Example 2: Find the vertical or horizontal asymptotes of each function.

a) $f(x) = \frac{1}{x+2}$

b) $g(x) = \frac{2x^2 - 11}{x^2 + 9}$

c) $h(x) = \frac{x^2 + 2x - 3}{x^2 + 5x - 6}$

Example 3: Finding End Behavior Models

Find the end behavior for

a) $f(x) = \frac{3x^4 + x^3 - 5x^2 + 4}{2x^2 - 3x + 5}$

b) $f(x) = \frac{2x^2 - 4x + 3}{5x^2 + 8}$

Example 4: Using Substitution.

a) $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) =$

b) $\lim_{x\to\infty}\left(2 + \frac{5}{x}\right)$

2.3: Continuity

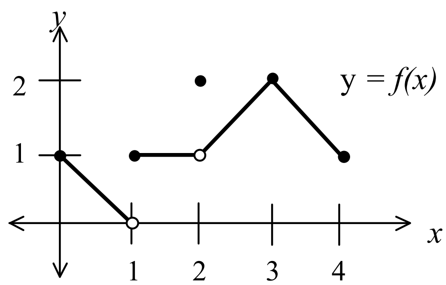
Continuity at a Point

Interior point: A function *f(x)* is continuous at an interior point *c* of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

Endpoint: A function *f(x)* is continuous at left endpoint a or its right endpoint *b* of its domain if

Example 1: Finding Continuity Graphically



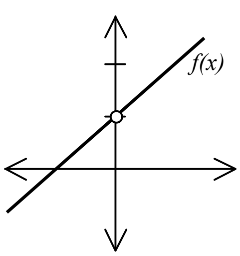
a) Where is f(x) continuous?

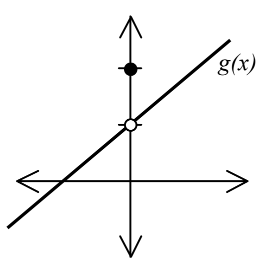
b) Not Continuous?

c) Compare this to the limits at these values.

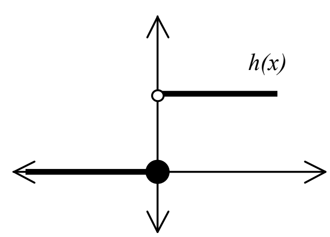
Types of Discontinuity

Removable Discontinuity:

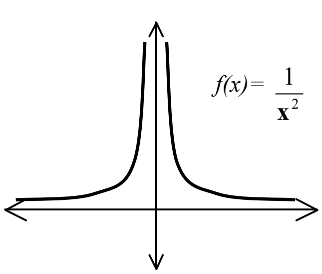




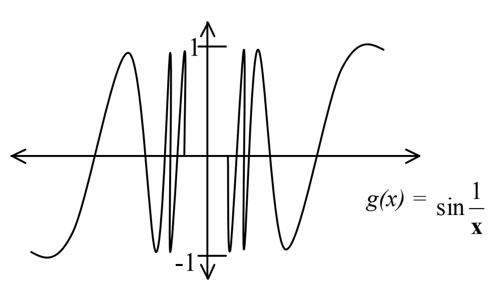
Jump discontinuity:



Infinite discontinuity:



Oscillating discontinuity:



Exploration 1: Let $f(x) = \dfrac{x^3 - 7x - 6}{x^2 - 9}$

1. Factor the denominator. What is the domain of *f*?

2. Investigate the graph of *f* around x = 3 to see that *f* has a removable discontinuity at x = 3.

3. How should *f* be defined at x = 3 to remove the discontinuity? Use zoom-in and tables as necessary.

4. Show that (x – 3) is a factor of the numerator of *f*, and remove all common factors. Now compute the limit as x $\rightarrow$ 3 of the reduced form for *f.*

5. Write the *extended function* so that it is continuous at x = 3.

$$g(x) = \begin{cases} & x \neq 3 \\ & x = 3 \end{cases}$$

**The function *g* is the <u>continuous extension</u> of the original function *f* to include x = 3.

Example 1: Determine if each function is continuous without graphing.

a) $f(x) = \begin{cases} 3x+2 & x<0 \\ x-4 & x \geq 0 \end{cases}$

b) $f(x) = \begin{cases} x^2-3 & x<-1 \\ x-1 & x \geq -1 \end{cases}$

c) $g(x) = \begin{cases} 3-x & x<2 \\ 2 & x=2 \\ \dfrac{x}{2} & x<2 \end{cases}$

Example 2: Determine the value for z so that the function is continuous.

a) $f(x) = \begin{cases} 2x+3 & x \leq 2 \\ ax+1 & x>2 \end{cases}$

b) $f(x) = \begin{cases} x^2+x+a & x<1 \\ x^3 & x \geq 1 \end{cases}$