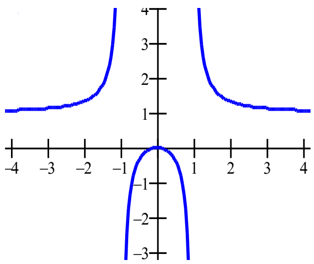
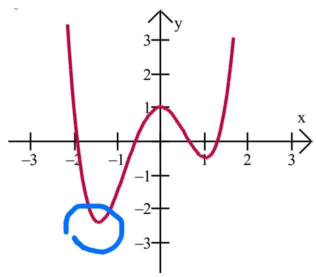
Boundedness

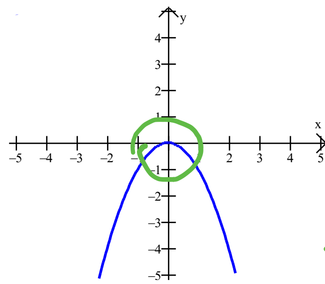
- A function *f* is **bounded below** if there is some number *b* that is less than or equal to every number in the range of *f*. Any such number *b* is called a lower bound of *f*.
- A function *f* is **bounded above** if there is some number *b* that is greater than or equal to every number in the range of *f*. Any such number is called an upper bound of *f*.
- A function is **bounded** if it is bounded both above *and* below.



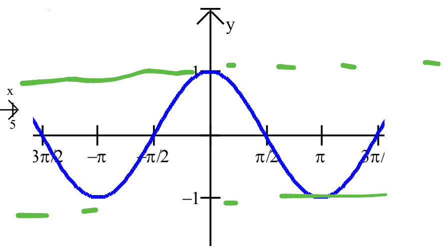
Not bounded above.
Not bounded below.



Not bounded above.
Bounded below.



Bounded above.
Not bounded below.

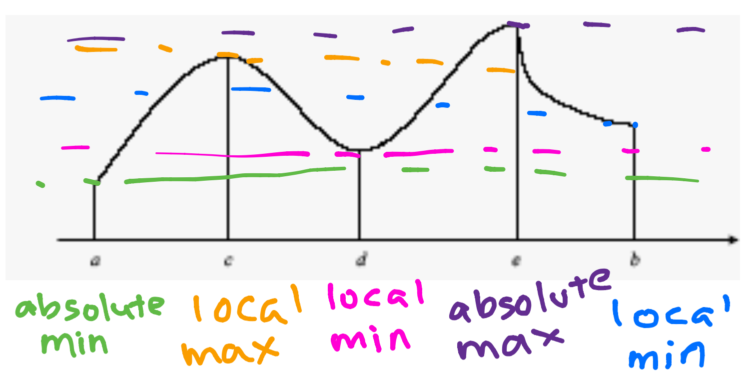


Bounded above.
Bounded below.

Local and Absolute Extrema

Extrema:	- peak (max) or a valley (min); - when a function changes from increasing to decreasing or from decreasing to increasing
Local extrema:	- relative max or min - peak or valley but not the highest/lowest point
Absolute extrema:	- the highest or lowest point on a function

Example 4: Identify whether the function above has a local or absolute extrema at each of the points *a*-*e*.

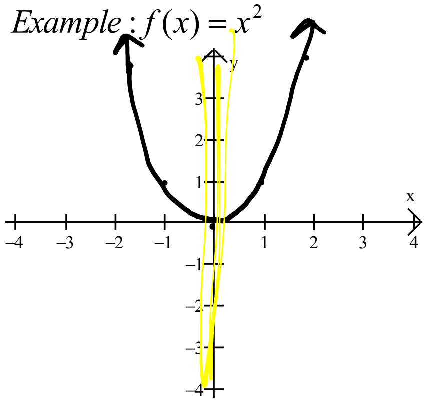


Symmetry:

Symmetric with respect to the y-axis - Even Function

Graphically

Example: $f(x) = x^2$



Numerically

x	*f(x)*
-3	9
-2	4
-1	1
1	1
2	4
3	9

Algebraically

$f(-x) = f(x)$

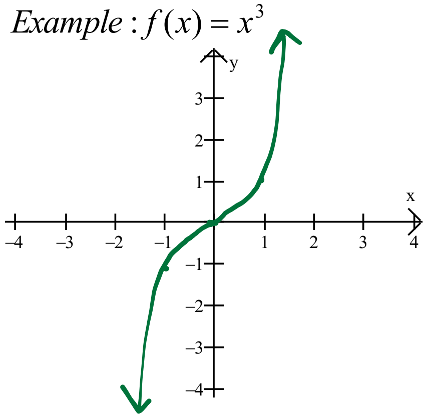
$f(x) = x^2$

$f(-x) = (-x)^2 = x^2$

Symmetric with respect to the origin - Odd Function

Graphically

Example: $f(x) = x^3$



Numerically

x	*f(x)*
-3	-27
-2	-8
-1	-1
1	1
2	8
3	27

Algebraically

$f(-x) = -f(x)$

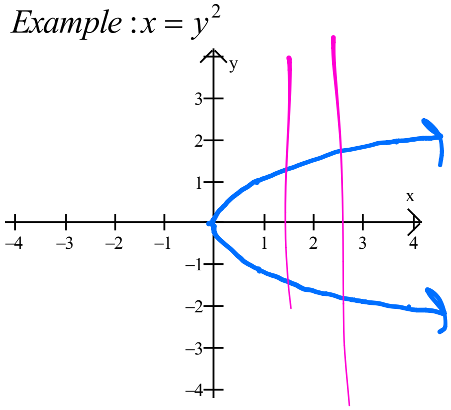
$f(x) = x^3$

$f(-x) = (-x)^3 = -x^3$

Symmetric with respect to the x-axis - Not a Function

Graphically

Example: $x = y^2$



Numerically

x	*f(x)*
9	3
4	2
1	1
1	-1
4	-2
9	-3

Algebraically

These graphs are not functions but they do have the following relation:

$(x, -y)$ is on the graph whenever (x, y)

Asymptotes

Horizontal Asymptote: y = b is a horizontal asymptote if and only if

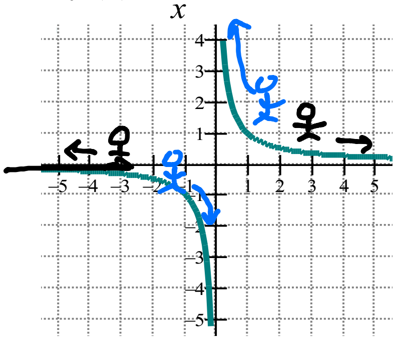
$$\lim_{x\to\pm\infty} f(x) = b$$

Vertical Asymptote: $x = a$ is a vertical asymptote on *f(x)* if and only if

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

Example 5: Identify any asymptotes on the graphs below.

a) $f(x) = \dfrac{1}{x}$



$\lim_{x\to\infty} f(x) = 0$

$\lim_{x\to-\infty} f(x) = 0$

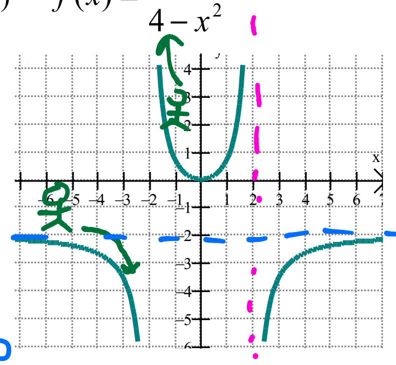
H.A. at $y = 0$

$\lim_{x\to 0^-} f(x) = -\infty$

$\lim_{x\to 0^+} f(x) = \infty$

V.A. at $x = 0$

b) $f(x) = \dfrac{2x^2}{4-x^2}$



$\lim_{x\to -2^-} f(x) = -\infty$

$\lim_{x\to -2^+} f(x) = \infty$

V.A. at $x = -2$

H.A. at $y = 2$

V.A. at $x = 2$

Limit notation:

$\lim_{x\to a} f(x)$
"Limit as x goes to a"

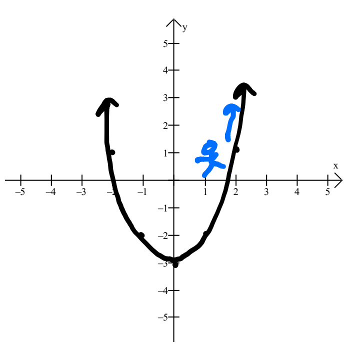
$\lim_{x\to a^+} f(x)$
"Limit as x goes to a from the positive side"

$\lim_{x\to a^-} f(x)$
"Limit as x goes to a from the negative side"

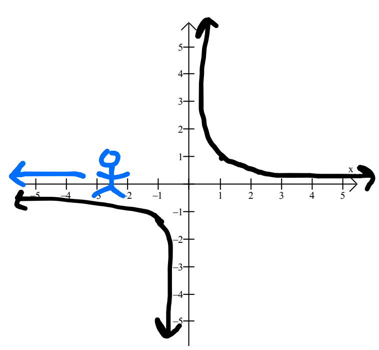
Limits to Infinity

Example 6: Find the limits graphically.

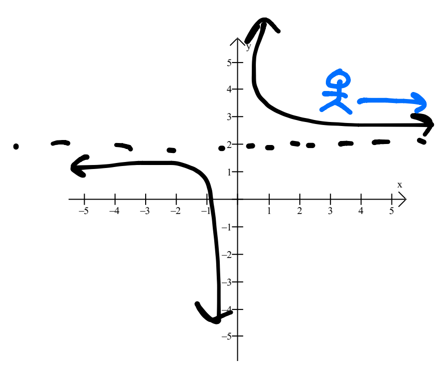
a) $\lim_{x\to\infty} x^2 - 3 = +\infty$



b) $\lim_{x\to-\infty} \dfrac{1}{x} = 0$



c) $\lim_{x\to\infty} \dfrac{1}{x} + 2 = 2$

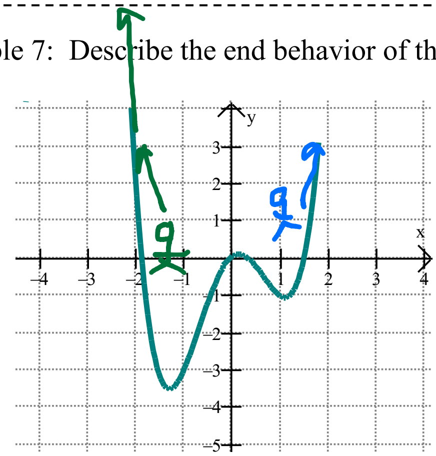


End Behavior

Right hand end behavior: $\lim_{x \to +\infty} f(x)$

Left hand end behavior: $\lim_{x \to -\infty} f(x)$

Example 7: Describe the end behavior of the graph below.

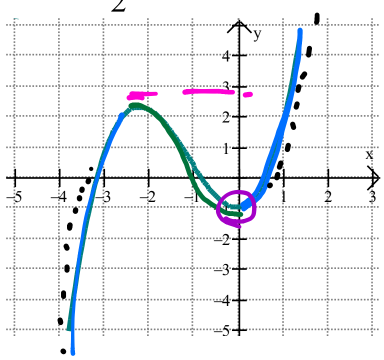


$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = \infty$

Example 6: Describe the behavior of the graph below:

a) $f(x) = \frac{1}{2}x^3 + 2x - 1$



Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Continuity: continuous

Increasing/Decreasing: increasing: $(-\infty, -2.3) \cup (0, \infty)$
decreasing: $(-2.3, 0)$

Symmetry: None

Boundedness: not bounded

Extrema: local max: 2.3 when $x = -2.3$
local min: -1 when $x = 0$

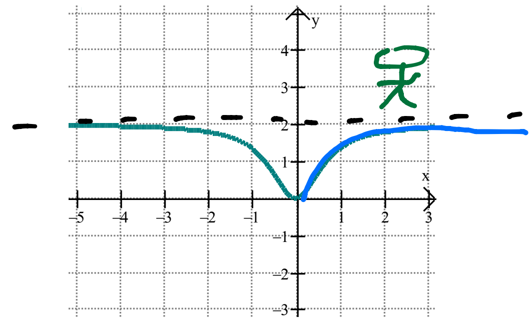
Asymptotes: None

End Behavior:

$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$

b) $f(x) = \frac{4x^2}{1 + 2x^2}$



Domain: $(-\infty, \infty)$

Range: $[0, 2)$

Continuity: continuous

Increasing/Decreasing: increasing: $(0, \infty)$
decreasing: $(-\infty, 0)$

Symmetry: about the y-axis

Boundedness: bounded

Extrema: absolute min: 0 when $x = 0$.

Asymptotes: $y = 2$

End Behavior:

$\lim_{x \to \infty} f(x) = 2$

$\lim_{x \to -\infty} f(x) = 2$