2.4: Real Zeros of Polynomial Functions

Long Division and the Division Algorithm

Example 1:

a) $32\overline{)527}$

16

-32

207

-192

15

16 R 15

$16+\frac{15}{32}$

b) $3x+2\overline{)3x^3+5x^2+8x+7}$

x^2+x+2

$-(3x^3+2x^2)$

$0 \quad 3x^2+8x$

$-(3x^2+2x)$

$0 \quad 6x+7$

$-(6x+4)$

$0 \quad 3$

$\boxed{x^2+x+2+\frac{3}{3x+2}}$

> **Division Algorithm for Polynomials**
>
> Let $f(x)$ *and* $d(x)$ be polynomials with the degree of f greater than or equal to the degree of d, and $d(x) \neq 0$. Then there are unique polynomials $q(x)$ and $r(x)$, called the **quotient and remainder,** such that
>
> $$f(x) = d(x) \cdot q(x) + r(x)$$
>
> where either $r(x) = 0$ or the degree of r is less than the degree of d

Using polynomial long division.

Example 2: Use long division to find the quotient and remainder.

a) $2x^2+x+1\overline{)2x^4-x^3+0x^2+0x-2}$

x^2-x

$-(2x^4+x^3+x^2)$

$0 \quad -2x^3-x^2+0x$

$+2x^3+x^2+x$

$0 \quad 0+x-2$

$\boxed{x^2-x+\frac{x-2}{2x^2+x+1}}$

Remainder Theorem

Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

Using the Remainder Theorem:

Example 3: Find the remainder when $f(x) = 3x^2 + 7x - 20$ is divided by

$f(2) = 6$

a) $x - 2$

$$\begin{array}{r} 3x + 13 \\ x-2\overline{)3x^2 + 7x - 20} \\ -3x^2 + 6x \quad \downarrow \\ \hline 0 \quad 13x - 20 \\ -13x + 26 \\ \hline 0 \quad 6 \end{array}$$

b) $x + 1$

$$\begin{array}{r|rrr} -1 & 3 & 7 & -20 \\ & \downarrow & -3 & -4 \\ \hline & 3 & 4 & -24 \end{array}$$

$f(-1) = -24$

c) $x + 4$

$$\begin{array}{r|rrr} -4 & 3 & 7 & -20 \\ & \downarrow & -12 & 20 \\ \hline & 3 & -5 & 0 \end{array}$$

$f(-4) = 0$

Factor Theorem

Let f be a polynomial function. Then $x - c$ is a factor if and only if $f(c) = 0$

Applying the Factor Theorem

Example 4: Divide.

$$\begin{array}{r} \boxed{3x - 5} \\ x+4\overline{)3x^2 + 7x - 20} \\ -3x^2 + 12x \quad \downarrow \\ \hline 0 \quad -5x - 20 \\ +5x + 20 \\ \hline 0 \quad 0 \end{array}$$

$$\boxed{3x^2 + 7x - 20 = (x+4)(3x-5)}$$

Fundamental Connections for Polynomial Functions

For a polynomial function f and a real number k, the following statements are equivalent:

1. $x = k$ is a solution (or root) of the equation $f(x) = 0$
2. k is zero of the function f.
3. k is an *x-intercept* of the graph of $y = f(x)$.
4. $x - k$ is a factor of $f(x)$.

Synthetic Division

Example 5: Divide using synthetic division

a) Divide $2x^3 - 3x^2 - 5x$ by $x - 3$

$x-3=0$
$x=3$

$$\begin{array}{r|rrrr} 3 & 2 & -3 & -5 & 0 \\ & & 6 & 9 & 12 \\ \hline & 2 & 3 & 4 & 12 \end{array}$$

$$\boxed{2x^2+3x+4+\frac{12}{x-3}}$$

$f(3)=12$

b) $\dfrac{2x^3 - 7x^2 + 4x - 5}{x-3}$

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 4 & -5 \\ & \downarrow & 6 & -3 & 3 \\ \hline & 2 & -1 & 1 & -2 \end{array}$$

$$\boxed{2x^2-x+1+\frac{-2}{x-3}}$$

c) $\dfrac{x^4 + 3x^3 + 8}{x+2}$

$$\begin{array}{r|rrrrr} -2 & 1 & 3 & 0 & 0 & 8 \\ & \downarrow & -2 & -2 & 4 & -8 \\ \hline & 1 & 1 & -2 & 4 & 0 \end{array}$$

$$\boxed{x^3+x^2-2x+4}$$

Rational Zeros Theorem

Let *f* be a polynomial function of degree 1 or higher of the form

$$f(x) = a_n x^n + a_{n-1}x^{x-1} + \cdots + a_1 x^1 + a_0 \qquad a_n \neq 0,\ \ a_0 \neq 0$$

Let p = factors of a_0 and Let q = factors of a_n

The list of possible rational zeros would be $\pm\dfrac{p}{q}$

Find the Rational Zeros

Example 6: Find the rational zeros of each polynomial.

a) $f(x) = x^3 + 5x^2 + 2x - 8$ $\quad p: -8$, $q: 1$

$\frac{p}{q}: \frac{\pm1, \pm2, \pm4, \pm8}{\pm1}$

possible rational zeros: $\pm1, \pm2, \pm4, \pm8$

$$\begin{array}{r|rrrr} 1 & 1 & 5 & 2 & -8 \\ & \downarrow & 1 & 6 & 8 \\ \hline & 1 & 6 & 8 & 0 \end{array}$$

$x^2 + 6x + 8$

$= (x+2)(x+4) = 0$

$x = -2 \quad x = -4$

$\boxed{\text{zeros: } x = 1,\ x = -2,\ x = -4}$

b) $f(x) = 3x^3 + 4x^2 - 5x - 2$ $\quad p: -2$, $q: 3$

$\frac{p}{q}: \frac{\pm1, \pm2}{\pm1, \pm3}$

possible rational zeros: $\pm1, \pm\frac{1}{3}, \pm2, \pm\frac{2}{3}$

$$\begin{array}{r|rrrr} 1 & 3 & 4 & -5 & -2 \\ & \downarrow & 3 & 7 & 2 \\ \hline & 3 & 7 & 2 & 0 \end{array}$$

$3x^2 + 7x + 2$

$$\begin{array}{r|rrr} -2 & 3 & 7 & 2 \\ & \downarrow & -6 & -2 \\ \hline & 3 & 1 & 0 \end{array}$$

$3x + 1 = 0$

$x = -\frac{1}{3}$

$\boxed{\text{Zeros: } x = -\tfrac{1}{3},\ x = -2,\ x = 1}$

c) $f(x) = x^4 - x^3 - 7x^2 + 13x - 6$ $\quad p: -6$, $q: 1$

$\frac{p}{q}: \frac{\pm1, \pm2, \pm3, \pm6}{\pm1}$

possible rational zeros: $\pm1, \pm2, \pm3, \pm6$

$$\begin{array}{r|rrrrr} 1 & 1 & -1 & -7 & 13 & -6 \\ & \downarrow & 1 & 0 & -7 & 6 \\ \hline & 1 & 0 & -7 & 6 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -7 & 6 \\ & \downarrow & 2 & 4 & -6 \\ \hline & 1 & 2 & -3 & 0 \end{array}$$

$x^2 + 2x - 3$

$= (x-1)(x+3) = 0$

$x = 1,\ x = -3$

$\boxed{\text{Zeros: } x = 1 \text{ mult. } 2,\ x = 2,\ x = -3}$

d) $f(x) = x^3 + 7x^2 + 13x + 6$ $\quad p: 6$, $q: 1$

$\frac{p}{q}: \frac{\pm1, \pm2, \pm3, \pm6}{\pm1}$

possible rational zeros: $\pm1, \pm2, \pm3, \pm4$

$$\begin{array}{r|rrrr} -2 & 1 & 7 & 13 & 6 \\ & \downarrow & -2 & -10 & -6 \\ \hline & 1 & 5 & 3 & 0 \end{array}$$

$x^2 + 5x + 3$