2.5: Complex Zeros and the Fundamental Theorem of Algebra

Fundamental Theorem of Algebra

A polynomial function of degree *n* has *n* complex zeros (real and imaginary). Some of these zeros may be repeated.

Fundamental Polynomial Connections in the Complex Case

The following statements about a polynomial function *f* are equivalent if *k* is a complex number

1. ________ is a solution (or root) of the equation ________
2. ______ is a zero of the function ______.
3. _______ is a factor of _________.

Exploring Fundamental Polynomial Connections

Example 1: Write the polynomial function in standard form, and identify the zeros of the function and the x-intercepts of its graph. Then sketch the graph.

a) The quadratic function

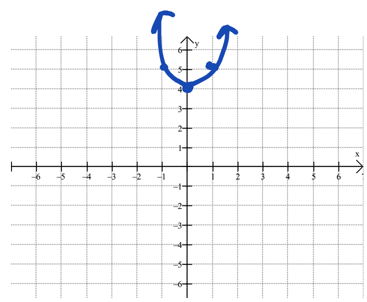
$f(x) = (x - 2i)(x + 2i)$

$x^2 - \cancel{2i + 2i} - 4i^2$

$x^2 - 4(-1) = \boxed{x^2 + 4}$

zeros: $x = -2i, 2i$

x-intercepts: none



b). The cubic function

$f(x) = (x - 5)(x - \sqrt{2}i)(x + \sqrt{2}i)$

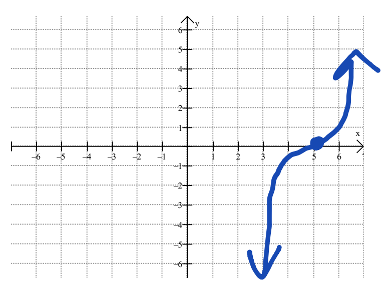
$f(x) = (x-5)(x^2 + \cancel{xi\sqrt{2}} - \cancel{xi\sqrt{2}} - \sqrt{4}i^2)$

$= (x-5)(x^2+2)$

$= x^3 - 5x^2 + 2x - 10$

zeros: $x = 5, i\sqrt{2}, -i\sqrt{2}$

x-intercepts: $(5,0)$



c). The quartic function

$f(x) = (x - 3)(x - 3)(x - i)(x + i)$

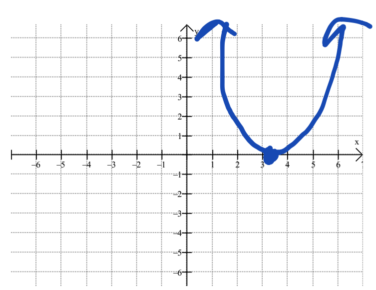
$f(x) = (x^2 - 6x + 9)(x^2 + 1)$

$= x^4 + x^2 - 6x^3 - 6x + 9x^2 + 9$

$= x^4 - 6x^3 + 10x^2 - 6x + 9$

zeros: $x = 3$ mult. 2, $i, -i$

x-intercepts: $(3,0)$



Complex Conjugate Zeros

Suppose that *f(x)* is a polynomial function with *real coefficients*. If a and b are real numbers with $b \neq 0$ and $a + bi$ is a zero of *f(x)*, then its complex conjugate $a - bi$ is also a zero of *f(x)*.

Finding a Polynomial from Given Zeros

Example 2: Write a polynomial function of minimum degree in standard form with real coefficients whose zeros include

a) -3, 4, and 2 – i, 2 + i

$$f(x) = (x+3)(x-4)(x-(2-i))(x-(2+i))$$

$$= (x^2 - x - 12)(x - 2 + i)(x - 2 - i)$$

$$= (x^2 - x - 12)(x^2 - 2x \cancel{-ix} - 2x + 4 \cancel{+2i} \cancel{+ix} \cancel{-2i} - \overset{+1}{i^2})$$

$$= (x^2 - x - 12)(x^2 - 4x + 5)$$

$$= x^4 - 4x^3 + 5x^2 - x^3 + 4x^2 - 5x - 12x^2 + 48x - 60$$

$$= \boxed{x^4 - 5x^3 - 3x^2 + 43x - 60}$$

Factoring a polynomial with Complex Zeros

Example 3: Find all zeros of $f(x) = x^5 - 3x^4 - 5x^3 + 5x^2 - 6x + 8$ and write *f(x)* in its linear factorization.

$\frac{p}{q}: \frac{\pm 1, \pm 2, \pm 4, \pm 8}{\pm 1}$ possible: $\pm 1, \pm 2, \pm 4, \pm 8$

$$\begin{array}{r|rrrrrr} 1 & 1 & -3 & -5 & 5 & -6 & 8 \\ & \downarrow & 1 & -2 & -7 & -2 & -8 \\ \hline & 1 & -2 & -7 & -2 & -8 & 0 \end{array}$$

$$\begin{array}{r|rrrrr} 4 & 1 & -2 & -7 & -2 & -8 \\ & \downarrow & 4 & 8 & 4 & 8 \\ \hline & 1 & 2 & 1 & 2 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 1 & 2 \\ & \downarrow & -2 & 0 & -2 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = \pm i$$

zeros: $1, 4, -2, i, -i$

Linear factorization: $f(x) = (x-1)(x-4)(x+2)(x-i)(x+i)$

Finding Complex Zeros

Example 4: The complex number z = 1 – 2i is a zero of $f(x) = 4x^4 + 17x^2 + 14x + 65$. Find the remaining zeros of *f(x)*.

1+2i

$(x-(1+2i))(x-(1-2i)) = (x-1-2i)(x-1+2i)$

$= x^2 - x + \cancel{2ix} - x + 1 - \cancel{2i} - \cancel{2ix} + \cancel{2i} - 4i^2$

$= x^2 - 2x + 5$

$$\begin{array}{r|l} & 4x^2 + 8x + 13 \\ x^2-2x+5 & 4x^4 + 0x^3 + 17x^2 + 14x + 65 \\ & -4x^4 + 8x^3 + 20x^2 \\ & 0 \quad 8x^3 - 3x^2 + 14x \\ & -8x^3 + 16x^2 + 40x \\ & 0 + 13x^2 - 26x + 65 \\ & 13x^2 - 26x + 65 \\ & 0 \end{array}$$

$x = \frac{-8 \pm \sqrt{64 - 4(4)(13)}}{2(4)}$

$x = \frac{-8 \pm \sqrt{-144}}{8}$

$x = \frac{-8 \pm 12i}{8}$

$\boxed{x = \frac{-2 \pm 3i}{2}}$

Factoring a Polynomial

Example 5: Write $f(x) = 3x^5 - 2x^4 + 6x^3 - 4x^2 - 9x + 6$ in factored form.

$\frac{p}{q}: \frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 3}$ possible: $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{3}, \pm \frac{2}{3}$

zeros: $1, \frac{2}{3}, -1, i\sqrt{3}, -i\sqrt{3}$

$$\begin{array}{r|rrrrrr} 1 & 3 & -2 & 6 & -4 & -9 & 6 \\ & \downarrow & 3 & 1 & 7 & 3 & -6 \\ \hline & 3 & 1 & 7 & 3 & -6 & 0 \end{array}$$

$$\begin{array}{r|rrrrr} 2/3 & 3 & 1 & 7 & 3 & -6 \\ & \downarrow & 2 & 2 & 6 & 6 \\ \hline & 3 & 3 & 9 & 9 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 3 & 3 & 9 & 9 \\ & \downarrow & -3 & 0 & -9 \\ \hline & 3 & 0 & 9 & 0 \end{array}$$

$3x^2 + 9 = 0$

$3x^2 = -9$

$x^2 = -3$

$x = \pm i\sqrt{3}$

factored form:

$f(x) = (x-1)(x - \frac{2}{3})(x+1)(x - i\sqrt{3})(x + i\sqrt{3})$

$\boxed{f(x) = (x-1)(3x-2)(x+1)(x - i\sqrt{3})(x + i\sqrt{3})}$