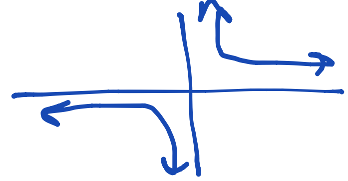
$y = \frac{1}{x}$



2.6: Graphs of Rational Functions

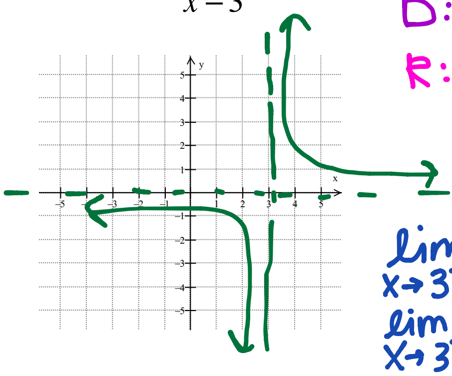
Rational Functions
Let *f* and *g* be polynomial functions with $g(x) \neq 0$. Then the function given by

$$r(x) = \frac{f(x)}{g(x)} \quad \text{is a } \textbf{rational function}$$

Example 1: Sketch the graph then find the domain of *f* and use limits to describe its behavior at values of x not in its domain using limits.

a) $f(x) = \dfrac{1}{x-3}$

D: $(-\infty, 3) \cup (3, \infty)$
R: $(-\infty, 0) \cup (0, \infty)$

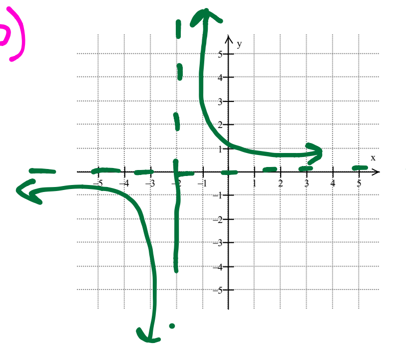


$\lim_{x \to 3^-} f(x) = -\infty$

$\lim_{x \to 3^+} f(x) = \infty$

b) $f(x) = \dfrac{1}{x+2}$

D: $(-\infty, -2) \cup (-2, \infty)$
R: $(-\infty, 0) \cup (0, \infty)$



$\lim_{x \to -2^-} f(x) = -\infty$

$\lim_{x \to -2^+} f(x) = \infty$

Horizontal and Vertical asymptotes:
In the graph of a function y = *f(x)*, the line y = b is a *horizontal asymptote* of the graph of *f* if

$$\lim_{x \to -\infty} f(x) = b \text{ or } \lim_{x \to \infty} f(x) = b$$

The line x = a is a *vertical asymptote* of the graph of *f* if

$$\lim_{x \to a^-} f(x) = \pm\infty \text{ or } \lim_{x \to a} f(x) = \pm\infty$$

Steps to Graph a Rational Function

The graph of $y = \dfrac{f(x)}{g(x)} = \dfrac{a_n x^n + \ldots + a_0}{b_m x^m + \ldots + b_0}$ has the following characteristics:

$\frac{\text{big}}{\text{big}}$

1. **End behavior asymptote:**
   If $n < m$, the end behavior asymptote is the horizontal asymptote $\mathbf{y = 0}$. bottom heavy

   If $n = m$, the end behavior asymptote is the horizontal asymptote $y = \dfrac{a_n}{b_m}$ equal

   If $n > m$, the end behavior asymptote is the *slant asymptote* which is the quotient polynomial function $y = q(x)$ where $f(x) = g(x)q(x) + r(x)$. There is no horizontal asymptote. top heavy
   LONG DIVISION
2. **Holes:**
   This occurs when you can factor the numerator and denominator and "cancel" something out.
3. **Vertical asymptotes:**
   These occur at the zeros of the denominator, provided that the zeros are not also zeros of the numerator of equal or greater multiplicity.
4. **X-intercepts:**
   These occur at the zeros of the numerator, which are not also zeros of the denominator.
5. **Y-intercept:**
   This is the value of *f(0)*, if defined.
6. **Plot Points:**

Example 2: Find the intercepts, asymptotes, use limits to describe the behavior at the vertical asymptotes, and analyze and draw the graph of the rational function.

a) $f(x) = \dfrac{x-1}{x^2 - x - 6} = \dfrac{(x-1)}{(x-3)(x+2)}$

1. $\frac{x}{x^2}$ bottom heavy $y=0$

2. No Holes

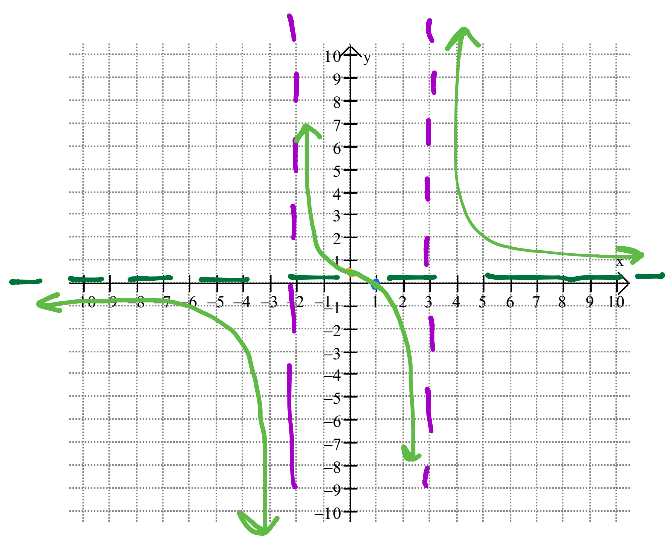
3. $(x-3)(x+2)=0$
   $x=3 \quad x=-2$

4. $x-1=0 \quad (1,0)$
   $x=1$

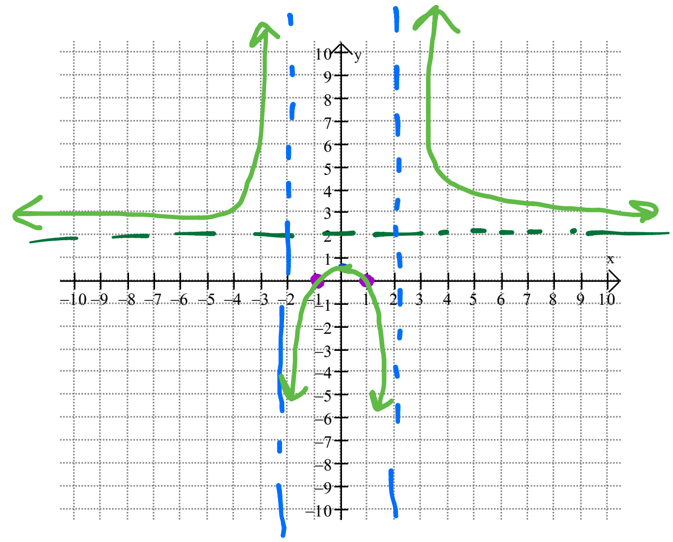
5. $f(0)=\dfrac{0-1}{0^2-0-6} = \dfrac{-1}{-6} = \dfrac{1}{6} \quad \left(0,\tfrac{1}{6}\right)$

6. $f(7)=\dfrac{7-1}{(7-3)(7+2)} = \dfrac{6}{(4)(9)} = +$

   $f(-4) = \dfrac{-4-1}{(-4-3)(-4+2)} = \dfrac{-5}{(-7)(-2)} = -$



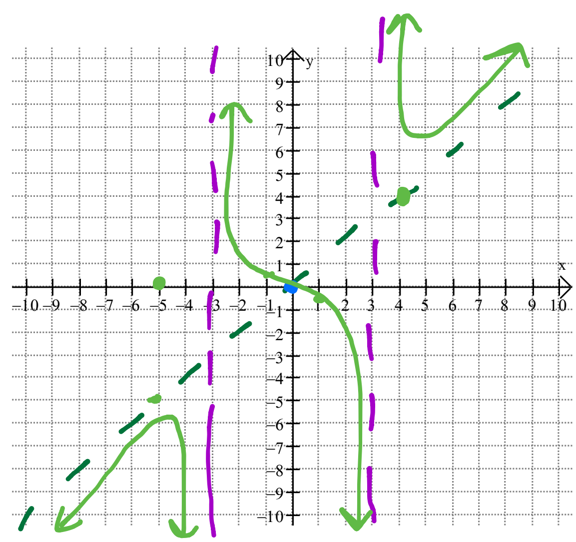
b) $f(x)=\frac{2x^2-2}{x^2-4}$ $\frac{2(x^2-1)}{x^2-4}=\frac{2(x+1)(x-1)}{(x-2)(x+2)}$



1. $\frac{2\cancel{x^2}}{\cancel{x^2}}$ equal $y=2$
2. No Holes
3. $(x-2)(x+2)=0$
   $x=2$ $x=-2$
4. $2(x-1)(x+1)=0$ $(1,0)$ & $(-1,0)$
   $x=1$ $x=-1$
5. $f(0)=\frac{2(0)^2-2}{0^2-4}=\frac{-2}{-4}=\frac{1}{2}$ $(0,\frac{1}{2})$
6. $f(-3)=\frac{2(-3+1)(-3-1)}{(-3-2)(-3+2)}=\frac{2(-2)(-4)}{(-5)(-1)}=+$

$f(4)=\frac{2(4)^2-2}{(4)^2-4}=\frac{32-2}{16-4}=+$

c) $f(x)=\frac{x^3}{x^2-9}=\frac{x^3}{(x-3)(x+3)}$



1. $\frac{x^3}{x^2}$ Top Heavy $y=x$

$$\begin{array}{r|l} & x \\ \hline x^2-9 & x^3+0x^2+0x+0 \\ & x^3 \quad\quad -9x \\ \hline & 0 \quad\quad -9x \end{array}$$

2. No Holes
3. $(x-3)(x+3)=0$
   $x=3$ $x=-3$
4. $x^3=0$ $(0,0)$
   $x=0$
5. $f(0)=\frac{0}{0^2-9}=\frac{0}{-9}=0$ $(0,0)$
6. $f(1)=\frac{1^3}{1^2-9}=\frac{1}{-8}$ $(1,-\frac{1}{8})$

$f(-1)=\frac{(-1)^3}{(-1)^2-9}=\frac{-1}{-8}$ $(-1,\frac{1}{8})$

$f(4)=\frac{4^3}{16-9}=\frac{64}{5}\approx 12.74$

$f(-5)=\frac{(-5)^3}{25-9}=\frac{-125}{16}\approx -7.8<-5$