5.2: Proving Trigonometric Identities

> **General Proof Strategy I**
> 1. The proof begins with the expression on one side of the identity.
> 2. The proof ends with the expression on the other side.
> 3. The proof in between consists of showing a sequence of expressions, each one easily seen to be equivalent to its preceding expression.

Example 1: **Prove the identity.**

$$\tan x + \cot x = \sec x \csc x$$

$$\tan x + \cot x = \frac{\sin x}{\sin x}\cdot\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\cdot\frac{\cos x}{\cos x} \qquad \text{LCD: } \sin x\cos x$$

$$\frac{\sin^2 x}{\sin x\cos x} + \frac{\cos^2 x}{\sin x\cos x} = \frac{\underbrace{\sin^2 x + \cos^2 x}_{=1}}{\sin x\cos x}$$

$$\frac{1}{\sin x\cos x} = \frac{1}{\sin x}\cdot\frac{1}{\cos x} = \csc x\sec x \quad \checkmark$$

> **General Proof Strategy II**
> 1. Begin with the more complicated expression and work toward the less complicated expression.
> 2. If no other move suggests itself, convert the entire expression to one involving sines and cosines.
> 3. Combine fractions by combing them over a common denominator.

Example 2: **Match the function** $f(x) = \dfrac{1}{\sec x - 1} + \dfrac{1}{\sec x + 1}$

with one of the following. Then confirm the match with a proof.

(i) $\boxed{2\cot x \csc x}$ (ii) $\dfrac{1}{\sec x}$

$$\frac{1}{\sec x - 1}\cdot\frac{(\sec x+1)}{(\sec x+1)} + \frac{1}{\sec x + 1}\cdot\frac{(\sec x-1)}{(\sec x-1)} \qquad \text{LCD: } (\sec x-1)(\sec x+1) = (\sec^2 x - 1)$$

$$\frac{\sec x+1}{\sec^2 x-1} + \frac{\sec x-1}{\sec^2 x-1} = \frac{2\sec x}{\sec^2 x - 1}$$

$$= \frac{2\sec x}{\tan^2 x} = \frac{2\left(\frac{1}{\cos x}\right)}{\left(\frac{\sin^2 x}{\cos^2 x}\right)} = 2\left(\frac{1}{\cancel{\cos x}}\right)\left(\frac{\cos^{\cancel{2}1} x}{\sin^2 x}\right) = 2\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin x}\right)$$

$$= 2\cot x\csc x \quad \checkmark$$

General Proof Strategy III

1. Use the algebraic identity $(a+b)(a-b)=a^2-b^2$ to set up applications of the Pythagorean identities.
2. Always be mindful of the "target" expression, and favor manipulations that bring you closer to your goal.

Example 3: **Setting up a Difference of Squares**

Prove the identity: $\dfrac{\cos t}{(1-\sin t)}=\dfrac{(1+\sin t)}{\cos t}$

$$\frac{(1+\sin t)}{\cos t}\cdot\frac{\cos t}{\cos t}=\frac{\cos t(1+\sin t)}{\cos^2 t}$$

$$=\frac{\cos t(1+\sin t)}{1-\sin^2 t}=\frac{\cos t\cancel{(1+\sin t)}}{(1-\sin t)\cancel{(1+\sin t)}}$$

$$=\frac{\cos t}{1-\sin t} \checkmark$$

Example 4: **Working from Both Sides**

Prove the identity: $\dfrac{\cot^2 u}{(1+\csc u)}=(\cot u)(\sec u-\tan u)$

$$\frac{\cot^2 u}{1+\csc u}=\frac{\csc^2 u-1}{1+\csc u}=\frac{(\csc u-1)\cancel{(\csc u+1)}}{\cancel{1+\csc u}}=\csc u-1$$

$$(\cot u)(\sec u-\tan u)=\left(\frac{\cos u}{\sin u}\right)\left(\frac{1}{\cos u}-\frac{\sin u}{\cos u}\right)=\frac{1}{\sin u}-1=\csc u-1 \checkmark$$

Example 5: **Prove an identity Useful in Calculus**

Prove the following identity: $\sin^2 x\cos^5 x=(\sin^2 x-2\sin^4 x+\sin^6 x)(\cos x)$

$$\sin^2 x\cos^5 x=\sin^2 x\cos^2 x\cos^2 x\cos x$$

$$=\sin^2 x(\cos^2 x)^2\cos x$$

$$=\sin^2 x(1-\sin^2 x)^2\cos x$$

$$=\sin^2 x(1-\sin^2 x)(1-\sin^2 x)\cos x$$

$$=\sin^2 x(1-2\sin^2 x+\sin^4 x)\cos x$$

$$=(\sin^2 x-2\sin^4 x+\sin^6 x)(\cos x) \checkmark$$

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p. 460: QR 1, 5, 6. 10

EX 2, 3, 5, 7. 9, 11, 14, 16, 18, 20, 22, 23.

24, 25, 28, 29, 33, 35, 39, 41, 53, 54, 57