NOTES: MATH 2 HONORS
Unit 10: Arc Length and Areas of Sectors

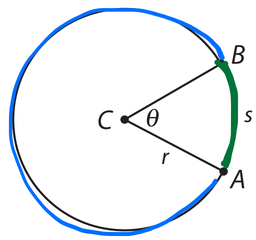
ARC LENGTH

So far, we have been referring to angle measures in degrees.
Another way to measure angles is with radians.

The **radian measure** of a central angle is defined as the ratio of the length of the arc intercepted by the angle to the radius of the circle.

Degrees: $s = 2\pi r\left(\frac{\theta}{360°}\right)$

Radians: $s = \cancel{2\pi} r\left(\frac{\theta}{\cancel{2\pi}}\right) = r\theta$

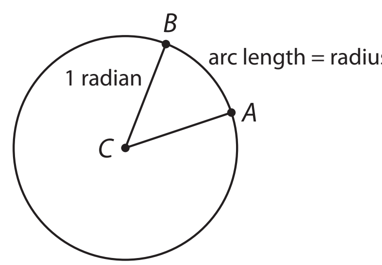


In the diagram, $\theta = \frac{s}{r}$

where θ is measured in radians.

When the intercepted arc is equal in length to the radius of the circle, the central angle measures 1 **radian**. This is how a radian is defined. So, if we consider a circle with a radius of 1 radian, then the distance around the circle is going to be 2π radians.

$C = 2\pi(1 \text{ radian}) = 2\pi \text{ radians}$



Since a circle contains 360° or 2π radians, we can convert between radian measure and degree measure using the following conversions:

- As a proportion: $\frac{radian\ measure}{2\pi} = \frac{degree\ measure}{360°}$
- To convert degrees to radians, multiply the angle by $\frac{\pi \text{ radians}}{180°}$
- To convert radians to degrees, multiply the angle by $\frac{180°}{\pi \text{ radians}}$

Example 1: Convert the following angles to radian measure. Leave your answer in terms of π.

a. an angle of 25°

$25 \cdot \left(\frac{\pi}{180}\right) = \frac{25\pi}{180} = \boxed{\frac{5\pi}{36}}$

b. an angle of 40°

$\frac{2\pi}{9}$

Example 2: Convert the following angles to degrees.

a. an angle of $\frac{\pi}{2}$ radians

$\frac{\cancel{\pi}}{2}\left(\frac{180}{\cancel{\pi}}\right) = \frac{180}{2} = \boxed{90°}$

a. an angle of $\frac{3\pi}{4}$ radians

$\frac{3\cancel{\pi}}{4}\left(\frac{180}{\cancel{\pi}}\right) = \frac{540°}{4} = \boxed{135°}$

Example 3: A circle has a radius of 4 units. Find the radian measure of a central angle that intercepts an arc of length 10.8 units.

$s = r\theta$

$$\frac{10.8}{4} = \frac{\cancel{4}(\theta)}{\cancel{4}}$$

$$\boxed{\theta = 2.7 \text{ radians}}$$

Example 4: A circle has a radius of 3.8 units. Find the length of an arc intercepted by a central angle measuring 2.1 radians.

$s = r\theta$

$$s = 3.8(2.1)$$

$$\boxed{s = 7.98 \text{ units}}$$

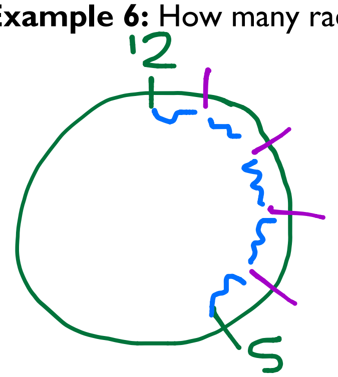
Example 5: A circle has a diameter of 20 feet. Find the length of an arc intercepted by a central angle measuring 36°.

$s = 2\pi r\left(\frac{\theta}{360}\right)$

$$s = 2\pi(10)\left(\frac{36^\circ}{360^\circ}\right) = 20\pi\left(\frac{1}{10}\right)$$

$$\boxed{s = 2\pi \text{ ft}}$$

Example 6: How many radians does the hour hand on a clock travel through from 12 to 5?



$$\frac{5}{12} \cdot 2\pi = \boxed{\frac{5\pi}{6} \text{ radians}}$$

AREA OF A SECTOR

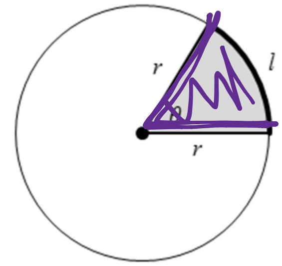
A region of a circle determined by two radii and the arc intercepted by the radii is called a **sector** of the circle. The shaded region on the right represents a **sector** of the circle. A sector is a fraction of a circle, so the ratio of the area of the sector to the area of the entire circle is equal to the measure of the central angle creating the sector to the measure of the entire circle.

$$\frac{\text{area of a sector}}{\text{area of circle}} = \frac{\text{measure of a central angle}}{\text{measure of circle}}$$

$$\frac{\text{area of a sector}}{\pi r^2} = \frac{\theta}{360°}$$

If the angle θ is in degrees, then the **area of a sector**, A, is $A = \frac{\pi\theta}{360°} \cdot r^2$

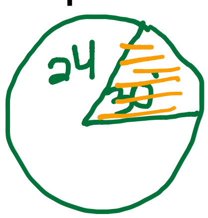
If the angle θ is in radians, then the **area of a sector**, A, is $A = \frac{1}{2} \cdot \theta r^2$



$A = \pi r^2\left(\frac{\theta}{360}\right)$

$A = \cancel{\pi} r^2\left(\frac{\theta}{2\cancel{\pi}}\right)$

Example 7: A circle has a radius of 24 units. Find the area of a sector with a central angle of 30°.



$A = \pi r^2\left(\frac{\theta}{360}\right)$

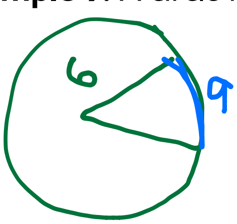
$= \pi(24^2)\left(\frac{30}{360}\right) = \boxed{48\pi \text{ units}^2}$

Example 8: A circle has a radius of 8 units. Find the area of a sector with a central angle of $\frac{3\pi}{4}$ radians.

$A = r^2\left(\frac{\theta}{2}\right)$

$A = (8^2)\left(\frac{\frac{3\pi}{4}}{2}\right) = 8^2 \cdot \frac{3\pi}{4} \cdot \frac{1}{2} = \boxed{24\pi \text{ units}^2}$

Example 9: A circle has a radius of 6 units. Find the area of a sector with an arc length of 9 units.



$s = r\theta$

$9 = 6\theta$

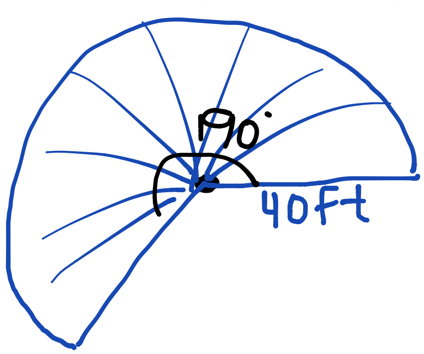
$\theta = \frac{9}{6} = \frac{3}{2}$ radians

$A = r^2\left(\frac{\theta}{2}\right)$

$= 6^2\left(\frac{3/2}{2}\right) = 6^2 \cdot \frac{3}{2} \cdot \frac{1}{2}$

$= \boxed{27 \text{ units}^2}$

Example 10: A rotating sprinkler sprays a stream of water 40 feet long. The sprinkler rotates 190°. What is the area of the portion of the yard that is watered by the sprinkler?



$A = \pi r^2\left(\frac{\theta}{360}\right)$

$= \frac{\pi}{1}(40^2)\left(\frac{190}{360}\right)$

$\boxed{A = \frac{7600\pi}{9} ft^2}$