NOTES: SECONDARY 2H
Unit 9: Sum/Differences Identities

Sum and Difference Identities

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\sin\beta\cos\alpha$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

Example 1: Find the exact value of $\sin 15°$ without using a calculator.

STEP #1: Rewrite the angle as a sum or difference statement using the angles we know (30°, 45°, 60°).

$\sin(45-30)$

STEP #2: Use the identities above to rewrite $\sin 15°$.

$\sin(45-30)=\sin 45°\cos 30°-\sin 30°\cos 45°$

STEP #3: Evaluate the above identities using the values on the table (you should have memorized).

$$\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}=\boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

Example 2: Find the exact value of the following using the sum and difference identities.

a. $\cos 75°$

$\cos(30+45)$

$\cos 30\cos 45-\sin 30\sin 45$

$$\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$$

$$=\boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

b. $\tan(-15°)$

$$\tan(30-45)=\frac{\tan 30-\tan 45}{1+\tan 30\tan 45}$$

$$\frac{3\left(\frac{\sqrt{3}}{3}-1\right)}{3\left(1+\frac{\sqrt{3}}{3}\cdot 1\right)}=\frac{\sqrt{3}-3}{3+\sqrt{3}}\cdot\frac{3-\sqrt{3}}{3-\sqrt{3}}$$

$$=\frac{3\sqrt{3}-3-9+3\sqrt{3}}{9-\cancel{3\sqrt{3}}+\cancel{3\sqrt{3}}-3}=\frac{-12+6\sqrt{3}}{6}=\boxed{-2+\sqrt{3}}$$

c. $\sin 105°$

$\sin(60+45)$

$\sin 60\cos 45+\sin 45\cos 60$

$$\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$$

$$=\boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

d. $\cos 120°$

$\cos(60+60)$

$=\cos 60\cos 60-\sin 60\sin 60$

$$\frac{1}{2}\cdot\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}=\frac{1}{4}-\frac{3}{4}=\frac{-2}{4}$$

$$=\boxed{-\frac{1}{2}}$$

We can also include all the obtuse angles we have found. (30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 210°, 225°, 240°, 270°, 300°, 315°, 330°, 360°)

Example 3: Find the exact value of the following using the sum and difference identities.

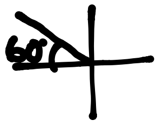
a. $\sin 165°$

$\sin(120+45)$

$\sin 120 \cos 45 + \sin 45 \cos 120$

$\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)$

$= \frac{\sqrt{6}}{4} + \frac{-\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$



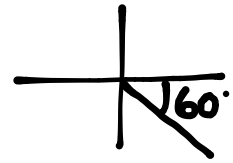
b. $\cos 255°$

$\cos(300-45)$

$\cos 300 \cos 45 + \sin 300 \sin 45$

$\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{-\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{2}}{4} + \frac{-\sqrt{6}}{4} = \boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$



c. $\tan 195°$

$\tan(150+45)$

$\frac{\tan 150 + \tan 45}{1 - \tan 150 \tan 45}$

$\frac{\left(-\frac{\sqrt{3}}{3} + 1\right) \cdot 3}{\left(1 - \left(-\frac{\sqrt{3}}{3}\right)\right) \cdot 3} = \frac{-\sqrt{3}+3}{3+\sqrt{3}} \cdot \frac{3-\sqrt{3}}{3-\sqrt{3}}$

$= \frac{-3\sqrt{3}+3+9-3\sqrt{3}}{9-3\sqrt{3}+3\sqrt{3}-3} = \frac{12-6\sqrt{3}}{6}$

$= \boxed{2-\sqrt{3}}$

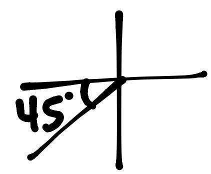
d. $\cos 285°$

$\cos(225+60)$

$\cos 225 \cos 60 - \sin 225 \sin 60$

$\left(\frac{-\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{-\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{-\sqrt{2}}{4} - \frac{-\sqrt{6}}{4} = \boxed{\frac{-\sqrt{2}+\sqrt{6}}{4}}$



Example 4: Using the trig identities, write the following expressions as the sine or cosine of an angle.

a. $\sin 22° \cos 13° + \cos 22° \sin 13°$

$\sin(22+13)$

$= \boxed{\sin(35°)}$

b. $\sin x° \sin 2x° - \cos x° \cos 2x°$

$-\cos x° \cos 2x° + \sin x° \sin 2x°$

$-(\cos x° \cos 2x° - \sin x° \sin 2x°)$

$-\cos(x+2x) = \boxed{-\cos(3x)}$

c. $\cos 60° \cos 45° + \sin 45° \sin 60°$

$\cos(60-45)$

$= \boxed{\cos(15°)}$